The Imagination Machine XVII

The Nabaala Theorem of General Subject-Relativity

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Abstract

We prove that the maximum order of self-classification available to any embedded epistemic system is a topological invariant of its observational boundary, determined entirely by the genus of that boundary. The result requires no metric, no physical assumptions, and no assumption about the topology of the containing manifold.

An embedded epistemic system compresses its observations into a quotient graph Q drawn on its observational boundary S. The depth of the simplicial tower above Q — the clique complex $X(Q)$ — measures the maximum order of relational self-classification the system can represent. We establish that this depth is bounded by $H(g) - 1$, where g is the genus of S and

$$H(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor$$

is the Heawood number. The bound is tight by the Ringel–Youngs theorem for $g \geq 1$ and by the Four Color Theorem for $g = 0$.

The nature of this constraint depends critically on the dimension of the observational boundary. For boundary dimension 1 the constraint is trivial; for boundary dimension ≥ 3 it vanishes entirely, since every finite graph embeds in S^3 without crossings. Boundary dimension 2 — the case of a three-dimensional observer — is the unique regime in which the genus of the boundary imposes a nontrivial, graduated categorical constraint on the simplicial tower. Three-dimensional observers are therefore epistemically special not by assumption but by the mathematics of surface embeddings.

We call the main result the *Nabaala Theorem of General Subject-Relativity*. The Nabaala Theorem of Subject-Relativity established in a companion paper is the special case $g = 0$, giving maximum tower depth $H(0) - 1 = 3$. The general theorem reveals that observers with higher-genus observational boundaries have categorically deeper towers, and that this difference is topological rather than physical: it cannot be overcome by any increase in information budget or computational resources.

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1 Introduction

How deeply can an epistemic system classify its own classifications? The question is not about computational power or memory capacity. It is about the structure of the surface through which the system encounters the world.

Any epistemic system embedded within an environment — one that models the world from inside it rather than surveying it from without — has access only to the observations that reach it through its observational boundary. Those observations carry relational structure: some things are similar, some different, some mutually related in higher-order ways. The system compresses this relational structure into a graph drawn on the boundary surface. The simplicial tower above that graph — its clique complex — measures how many orders of relational self-reference the system can represent: it can classify observations (order 0), classify relations between observations (order 1), classify relations between relations (order 2), and so on.

The central question is: how high can this tower go?

The answer, we show, depends on the topology of the observational boundary, and on nothing else. Specifically, it depends on the genus — the number of handles — of the boundary surface. The Heawood bound, a classical result of combinatorial topology, establishes that any graph drawn on a surface of genus g has chromatic number at most $H(g)$, where $H(g) = \lfloor (7 + \sqrt{1 + 48g})/2 \rfloor$. Since the clique number of a graph never exceeds its chromatic number, the clique complex of the quotient graph has dimension at most $H(g) - 1$. The tower terminates at depth $H(g) - 1$.

This is the Nabaala Theorem of General Subject-Relativity. It is general because it holds for any compact orientable observational boundary of genus g, without assuming a specific topology for the boundary or for the containing manifold, and without any physical assumptions whatsoever. It is subject-relative because the categorical frame — the maximum tower depth — varies across observers with different boundary genera: a sphere-bounded observer has depth at most three; a torus-bounded observer has depth at most six. This variation is not empirical. It is topological.

The theorem also identifies the unique epistemic role of three-dimensional observers. An observer of dimension k has an observational boundary of dimension $k - 1$. For $k - 1 = 1$ (a two-dimensional observer) the constraint is trivial. For $k - 1 \geq 3$ (a four-dimensional or higher observer) every finite graph embeds on the boundary without crossings and the topological constraint vanishes. Only for $k - 1 = 2$ — a three-dimensional observer with a two-dimensional boundary — does the genus of the boundary impose a nontrivial, graduated constraint. Three-dimensional observers occupy the unique dimensionally special regime in which surface topology is maximally epistemically informative.

The paper is self-contained. Section 2 introduces the minimal formal framework. Section 3 analyzes the three regimes of boundary dimension. Section 4 recalls the Heawood bound and Ringel–Youngs theorem. Section 5 states and proves the Nabaala Theorem of General Subject-Relativity. Section 6 develops the ladder of self-classification. Section 7 situates the result within the Imagination Machine series. Section 8 discusses open questions.

2 The Minimal Formal Framework

We introduce the framework in its minimal form, sufficient for the present paper. Readers familiar with the Imagination Machine series will recognize these as special cases of the fuller apparatus developed there; readers new to the series will find the definitions self-contained.

Definition 2.1 (Embedded Epistemic System)**.** An *embedded epistemic system* is a triple (D, W, ω) where:

- D is a finite set of *observations*;
- W is a set of *world models*;
- $\omega\colon D \to Z$ is a *classifier* that partitions D into equivalence classes, for some finite set Z.

The classifier induces an equivalence relation $d_1 \sim d_2$ iff $\omega(d_1) = \omega(d_2)$, and a *quotient graph* $Q = D/{\sim}$ whose vertices are the equivalence classes and whose edges connect classes that are observationally adjacent.

Definition 2.2 (Observational Boundary)**.** An embedded epistemic system of dimension k has an *observational boundary* S: the $(k-1)$-dimensional surface through which all observations reach the system. The quotient graph Q is drawn on S.

Definition 2.3 (Simplicial Tower)**.** The *clique complex* $X(Q)$ of the quotient graph Q is the simplicial complex whose j-simplices are the $(j+1)$-cliques of Q. The *simplicial tower* is the sequence of skeleta $X(Q)^{(0)} \subseteq X(Q)^{(1)} \subseteq \cdots \subseteq X(Q)$. The *tower depth* is $\dim X(Q) = \omega(Q) - 1$, where $\omega(Q)$ is the clique number of Q.

Remark 2.4 (Interpretation of Tower Depth)*.* The tower depth measures the maximum order of relational self-classification the system can represent. At depth 0, the system classifies observations. At depth 1, it classifies pairs of observations — binary relations. At depth 2, it classifies triadic relational structures. At depth k, it classifies $(k+1)$-way mutual relations among observations. The tower depth is therefore the maximum order of self-reference available within the system's representational architecture.

Assumption 2.5 (Observational Boundary as Compact Orientable Surface)**.** The observational boundary S is a compact orientable surface of genus $g \geq 0$, and the quotient graph Q is a finite graph drawn on S.

Assumption 2.5 is the only geometric input required by the theorem. It states that the relational structure of observations lives on a surface — a two-dimensional boundary — and that this surface has a well-defined genus. No metric on S, no specific topology for the containing manifold, and no physical assumptions are required.

3 The Three Regimes of Boundary Dimension

Before stating the main theorem, we identify the three qualitatively distinct regimes that arise as the dimension of the observational boundary varies. This analysis motivates Assumption 2.5 and clarifies why two-dimensional boundaries are the unique epistemically interesting case.

Proposition 3.1 (Three Regimes)**.** *Let the observational boundary S have dimension $d = k - 1$, where k is the dimension of the observer.*

(i) $d = 1$*: trivial constraint. Every finite graph on S^1 is a subgraph of a cycle, hence planar. The clique number satisfies $\omega(Q) \leq 2$, giving tower depth at most 1. The topological constraint is present but trivially small.*

(ii) $d = 2$*: nontrivial, genus-dependent constraint. The chromatic number of any finite graph on a compact surface of genus g is bounded by $H(g)$ (the Heawood bound). This gives tower depth at most $H(g) - 1$, a quantity that varies nontrivially with g and is tight. This is the regime of the present theorem.*

(iii) $d \geq 3$*: constraint vanishes. Every finite graph embeds in $\mathbb{R}^3$ without crossings [3], and therefore in S^3 by one-point compactification. No chromatic or clique bound follows from the topology of the boundary alone. Only physical constraints (e.g. information-theoretic bounds) can limit the tower.*

Proof. (i) A graph on S^1 uses arcs of the circle as edges; any such graph is a subgraph of a cycle, which has clique number 2.

(ii) This is the content of Sections 4 and 5.

(iii) The classical result that every finite graph has a straight-line embedding in $\mathbb{R}^3$ follows from the fact that vertices can be placed on the moment curve (t, t^2, t^3) and edges drawn as straight lines; no two such edges cross [3]. The one-point compactification of $\mathbb{R}^3$ is S^3, giving the embedding in S^3. □

Remark 3.2 (The Special Status of Three-Dimensional Observers). Proposition 3.1 identifies boundary dimension 2 — equivalently, observer dimension 3 — as the unique regime in which the genus of the observational boundary imposes a nontrivial, graduated, and tight categorical constraint on the tower. Below this dimension the constraint is present but trivially small; above it the constraint vanishes entirely. Three-dimensional observers are therefore epistemically special not by assumption but by the mathematics of graph embeddings in surfaces.

4 The Heawood Bound and the Ringel–Youngs Theorem

We recall the classical results on graph colorings on surfaces that underlie the main theorem.

Definition 4.1 (Heawood Number)**.** For $g \geq 1$, the *Heawood number* is

$$H(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor . \tag{1}$$

For $g = 0$ we set $H(0) = 4$, consistent with the Four Color Theorem.

Theorem 4.2 (Heawood Bound [4])**.** *For any finite graph G embedded on a compact orientable surface of genus $g \geq 1$, the chromatic number satisfies $\chi(G) \leq H(g)$.*

Proof sketch. For a connected graph embedded on a surface of genus g, the generalized Euler formula gives $V - E + F = 2 - 2g$. Since each face is bounded by at least three edges and each edge borders at most two faces, $3F \leq 2E$, giving $E \leq 3(V - 2 + 2g) = 3V - 6 + 6g$. The average degree satisfies

$$\bar{d} = \frac{2E}{V} \leq 6 - \frac{12}{V} + \frac{12g}{V} < 6 + \frac{12g}{V - 1}.$$

For large V this is less than $H(g)$, so some vertex has degree less than $H(g)$. A greedy coloring argument on the graph with that vertex removed (inducting on V) gives $\chi(G) \leq H(g)$. □

Theorem 4.3 (Four Color Theorem [1])**.** *For $g = 0$: every finite planar graph satisfies $\chi(G) \leq 4 = H(0)$.*

Theorem 4.4 (Ringel–Youngs [5])**.** *For every $g \geq 1$, the complete graph $K_{H(g)}$ embeds on the compact orientable surface of genus g. Consequently the Heawood bound is tight: for each $g \geq 1$ there exist graphs on surfaces of genus g requiring exactly $H(g)$ colors.*

Remark 4.5. The Heawood bound and the Ringel–Youngs theorem together give a complete and tight characterization of the chromatic number of graphs on compact orientable surfaces, for all $g \geq 0$. The case $g = 0$ is the Four Color Theorem; the cases $g \geq 1$ are the Ringel–Youngs theorem.

The Heawood number grows with g:

Genus g	$H(g)$	Surface
0	4	Sphere S^2
1	7	Torus
2	8	Double torus
3	9	Triple torus
4	9	
5	10	
6	10	
7	11	

5 The Nabaala Theorem of General Subject-Relativity

Lemma 5.1 (Clique Number Bounded by Chromatic Number)**.** *For any finite graph G, $\omega(G) \leq \chi(G)$.*

Proof. Any proper coloring assigns distinct colors to all vertices of a clique, so the number of colors used is at least the size of the largest clique. □

Theorem 5.2 (Nabaala Theorem of General Subject-Relativity)**.** *Let an embedded epistemic system satisfy Assumption 2.5: its observational boundary S is a compact orientable surface of genus $g \geq 0$, and the quotient graph Q is a finite graph drawn on S. Then:*

(i) Categorical tower termination. The simplicial tower terminates at depth at most $H(g) - 1$:

$$\dim X(Q) \leq H(g) - 1.$$

This bound follows from the topology of S alone, without physical assumption.

(ii) Tightness. The bound $H(g) - 1$ is achieved: for each $g \geq 0$ there exist quotient graphs Q on surfaces of genus g whose tower reaches exactly depth $H(g) - 1$.

(iii) General subject-relativity. The maximum tower depth $H(g) - 1$ is a topological invariant of the observational boundary. Observers with observational boundaries of different genera have categorically different maximum tower depths. The categorical frame — the ceiling on self-classification — is itself subject-relative, varying by genus. This variation is topological, not physical: it cannot be overcome by any increase in information budget or computational resources.

(iv) The Nabaala Theorem of Subject-Relativity as special case. For $g = 0$, $H(0) - 1 = 3$, recovering the categorical bound of the Nabaala Theorem of Subject-Relativity [11].

Proof. (i) By Theorems 4.2 and 4.3, $\chi(Q) \leq H(g)$. By Lemma 5.1, $\omega(Q) \leq \chi(Q) \leq H(g)$. The dimension of the clique complex $X(Q)$ equals $\omega(Q) - 1 \leq H(g) - 1$. The k-skeleton $X(Q)^{(k)}$ is empty for $k > H(g) - 1$.

(ii) For $g \geq 1$: by Theorem 4.4, $K_{H(g)}$ embeds on a surface of genus g. Its clique complex has dimension $H(g) - 1$. For $g = 0$: K_4 is planar and has clique complex of dimension $3 = H(0) - 1$.

(iii) Since $H(g)$ depends only on g, and g is a topological invariant of S (invariant under homeomorphism), the bound $H(g) - 1$ is a topological invariant of the boundary. Two observers whose boundaries have different genera $g \neq g'$ have $H(g) \neq H(g')$ whenever H is injective at those values, giving categorically different tower depths. Since H is non-decreasing and the differences are topological rather than metric or physical, no physical resource can bridge the gap.

(iv) Setting $g = 0$: $H(0) = 4$, so $H(0) - 1 = 3$. □

Remark 5.3 (What General Subject-Relativity Means). The Nabaala Theorem of Subject-Relativity [11] identified two levels of subject-relativity. Categorically, all observers with $g = 0$ boundaries share the same tower ceiling of three. Subject-relatively, within that ceiling, the Bekenstein bound locates each observer at a specific depth determined by its surface area. The present theorem reveals a third, deeper level: the ceiling itself varies by genus. An observer with a $g = 1$ (toroidal) boundary has a categorical ceiling of six, not three. No amount of physical resources available to a $g = 0$ observer can raise its ceiling to six; the difference is written in the topology of the boundary, not in the physics of the observer.

Remark 5.4 (Mathematics Implies, Topology Differentiates, Physics Instantiates). The results across the series establish a three-level structure of necessity. Mathematics implies the existence of a categorical frame for any embedded observer — the tower must terminate at some finite depth. Topology differentiates the categorical frames across observers — the genus of the boundary determines which frame applies. Physics instantiates each specific observer within its topologically determined frame — the Bekenstein bound locates the observer at depth $K(A)$ within the ceiling $H(g) - 1$. The present theorem operates at the second level; the Nabaala Theorem of Subject-Relativity operates at the third.

6 The Ladder of Self-Classification

Definition 6.1 (Ladder of Self-Classification)**.** The *ladder of self-classification* is the sequence

$$d(g) = H(g) - 1, \quad g = 0, 1, 2, \ldots,$$

giving the maximum order of self-classification available to any embedded observer with observational boundary of genus g.

Genus g	$H(g)$	Tower depth $d(g)$	Surface
0	4	3	Sphere
1	7	6	Torus
2	8	7	Double torus
3	9	8	Triple torus
4	9	8	
5	10	9	
6	10	9	
7	11	10	

Remark 6.2 (Equivalence Classes on the Ladder). The function $g \mapsto d(g)$ is non-decreasing but not injective: some consecutive values of g give the same tower depth (for example, $d(3) = d(4) = 8$). This means there are equivalence classes of observational boundary genera that are categorically indistinguishable in terms of self-classification depth. The epistemically relevant partition of surfaces is coarser than the topological classification by genus.

Remark 6.3 (Our Position on the Ladder)*.* Three-dimensional observers with spherical ($g = 0$) observational boundaries occupy the bottom rung: maximum tower depth three. An observer with a toroidal ($g = 1$) observational boundary would have access to six orders of self-classification — categorically more, not merely physically more. The jump from rung 0 to rung 1 of the ladder cannot be bridged by any physical resource; it requires a different topology of the observational boundary.

7 Relation to the Imagination Machine Series

The present paper is part of the Imagination Machine series, which develops a formal framework for embedded epistemic systems across eighteen papers. We briefly situate the main result within that series for readers approaching from it; readers new to the series will find the present paper self-contained.

The series establishes in earlier papers that the observations of an embedded epistemic system are compressed by a classifier into a quotient graph Q, and that the clique complex of this graph realizes the simplicial tower of representational depth [6, 7, 8]. The geometric papers of the series argue that the observational boundary of a three-dimensional observer embedded in a four-dimensional containing manifold is homeomorphic to S^2 [9, 10], the case $g = 0$ of the present theorem.

The Nabaala Theorem of Subject-Relativity [11] established the $g = 0$ special case of the present result by two arguments: categorically, from Kuratowski's theorem (no K_5 in a planar graph, so tower depth ≤ 3); and subject-relatively, from the Bekenstein bound (the information budget of the surface determines the accessible depth within that categorical ceiling). The present paper generalizes the categorical argument to arbitrary genus without assuming any specific geometry.

The principle that emerges across the final papers of the series — that mathematics implies the categorical frame, topology differentiates the frames, and physics instantiates observers within them — is stated most generally here.

8 Discussion

The Nabaala Theorem of General Subject-Relativity establishes that the maximum order of self-classification available to any embedded epistemic system is a topological invariant of its observational boundary. The proof uses only the Heawood bound, the Ringel–Youngs theorem, and the Four Color Theorem, together with the minimal formal framework of Section 2. No metric, no physics, and no assumption about the topology of the containing manifold is required.

Several questions remain open.

Non-orientable surfaces. The theorem is stated for compact orientable surfaces. For non-orientable surfaces (classified by crosscap number k), the chromatic bound is $\lfloor (7 + \sqrt{1 + 24k})/2 \rfloor$ [2]. A version of the theorem for non-orientable boundaries would complete the classification.

Physical realization of higher-genus boundaries. The theorem establishes the categorical consequence of a genus-g boundary. What physical or geometric conditions would produce a toroidal or higher-genus observational boundary — what kind of observer or environment this would require — is not addressed here and is an open question for future work.

Quantum topology. The observational boundary has been treated as a classical topological object with a fixed genus. In a quantum-gravitational setting the topology of the boundary may fluctuate. A version of the theorem in which the genus is a quantum observable, distributed across the rungs of the ladder, would give a quantum theory of the categorical frame of self-classification.

The dimensional analysis beyond dimension two. Proposition 3.1 identifies three regimes of boundary dimension. The regime $d \geq 3$ is characterized only negatively: the topological constraint vanishes. A positive characterization of the constraints that do apply in higher dimensions — presumably information-theoretic or physical rather than topological — is an open question.

The epistemic invariant of a surface. The function $g \mapsto d(g) = H(g) - 1$ is non-decreasing but not injective: distinct genera can give the same tower depth, so the epistemically relevant partition of surfaces is strictly coarser than the topological classification by genus. This raises the question of what the correct epistemic invariant of a surface is — a quantity that captures exactly the information in $H(g)$ without the redundancy of genus. Whether this invariant has a direct topological or combinatorial characterization, independent of the detour through chromatic number, is an open question. Its identification would give the Nabaala Theorem its sharpest possible form: not "the tower depth is bounded by a function of the genus" but "the tower depth is a topological invariant, and here it is."

A topological cognitive bound on artificial epistemic systems. The theorem applies to any embedded epistemic system satisfying Assumption 2.5 — any system that models the world through a quotient graph on a two-dimensional observational boundary. This includes artificial systems. If a sufficiently general artificial intelligence is embedded in the world rather than surveying it from without, and if its observational boundary is two-dimensional and spherical, then the Nabaala Theorem of Subject-Relativity applies: its maximum order of self-classification is three, categorically, regardless of computational resources, training data, or architectural scale. The Bekenstein bound then further constrains the accessible depth within that ceiling as a function of the system's physical surface area.

This is a topological cognitive upper bound on embedded artificial intelligence, not a limitation of any particular implementation but a structural consequence of embeddedness itself. It reframes a central question in the theory of artificial general intelligence: not how to make embedded systems more powerful within a fixed architecture, but what topology of observational boundary would be required to access deeper orders of self-classification. A system with a toroidal ($g = 1$) observational boundary would have a categorical ceiling of six rather than three — categorically deeper, not merely physically larger. The question of how to physically realize a higher-genus observational boundary for an artificial system is therefore not an engineering question but a topological one, and the ladder of self-classification given in Section 6 provides the precise targets.

The ladder therefore has implications not only for the theory of knowledge but for the practice of building minds. Three-dimensional observers occupy the bottom rung — and it is from that rung that we reach upward toward the agents we create.

The series began with a single constraint: an embedded epistemic system can at most classify the ways in which it classifies the world, within the world itself. The Nabaala Theorem of General Subject-Relativity gives this constraint its most general mathematical expression. The maximum order of self-classification is not determined by the resources of the observer, nor by the physics of its environment, but by the topology of the surface through which it looks. It is written in the shape of the boundary between the observer and the world — and that shape, the theorem says, is all that matters.

References

[1] K. Appel and W. Haken. Every planar map is four colorable. *Illinois Journal of Mathematics*, 21(3):429–490, 1976.

[2] P. Franklin. A six colour problem. *Journal of Mathematics and Physics*, 13:363–369, 1934.

[3] I. Fáry. On straight line representation of planar graphs. *Acta Scientiarum Mathematicarum*, 11:229–233, 1948.

[4] P. J. Heawood. Map-colour theorem. *Quarterly Journal of Pure and Applied Mathematics*, 24:332–338, 1890.

[5] G. Ringel and J. W. T. Youngs. Solution of the Heawood map-coloring problem. *Proceedings of the National Academy of Sciences*, 60(2):438–445, 1968.

[6] M. Tracy. The Imagination Machine I: A View from Somewhere. Unpublished manuscript, Boston University, 2025.

[7] M. Tracy. The Imagination Machine X: The Simplicial Structure of Compression and Extension. Unpublished manuscript, Boston University, 2026.

[8] M. Tracy and S. T. Nabaala. The Imagination Machine XI: Graph-Theoretic Realizations of Compression and Extension. Unpublished manuscript, Boston University, 2026.

[9] M. Tracy. The Imagination Machine XIV: The View from Nowhere and the Center of the Hypersphere. Unpublished manuscript, Boston University, 2026.

[10] M. Tracy and S. T. Nabaala. The Imagination Machine XV: Chromatic Number and the Sensory Constraint on Embedded Observers. Unpublished manuscript, Boston University, 2026.

[11] M. Tracy and S. T. Nabaala. The Imagination Machine XVI: The Bekenstein Bound, Tower Termination, and the Physical Grounding of Epistemic Closure. Unpublished manuscript, Boston University, 2026.