

The Imagination Machine XVIII

The Bubble Bursts:

The Periodic Table, the Hydrogen Atom,
and the Geometry of the Containing Manifold

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Abstract

The Imagination Machine series established that the observational surface of any embedded epistemic system is a two-sphere S^2 , that the containing manifold is the three-sphere S^3 sourced by the $k = +1$ Friedmann–Robertson–Walker solution to Einstein’s field equations, and that the Nabaala Theorem of General Subject-Relativity bounds the maximum order of self-classification by the Heawood number of the observational boundary’s genus. The present paper identifies a closing loop.

In 1935, Vladimir Fock showed that the hydrogen atom in three-dimensional momentum space is equivalent to a free particle moving on the three-sphere S^3 [1]. The “accidental” degeneracy of hydrogen’s energy levels — the fact that states with different angular momentum l share the same energy — is not accidental. It is the natural consequence of the $SO(4)$ symmetry of a free particle on S^3 . The degeneracy of the n -th energy level is n^2 without spin and $2n^2$ with spin, giving the sequence 2, 8, 18, 32, . . . electrons per shell. This is the structure of the periodic table.

The three-sphere that Fock identified in momentum space is the same S^3 that the series identified as the containing manifold of the embedded observer, sourced by the same Einstein field equations. The angular part of the $SO(4)$ representations on S^3 restricts to $SO(3)$ representations on S^2 — the spherical harmonics — whose chromatic structure is bounded by the Four Color Theorem, the $g = 0$ special case of the Nabaala Theorem.

The loop therefore closes as follows. Einstein’s field equations source the $k = +1$ FRW geometry, which gives S^3 as the containing manifold. From S^3 two consequences follow by independent routes. The first route, through the Bekenstein bound and the Nabaala Theorem, gives the topological bound on self-classification for embedded epistemic systems. The second route, through Fock’s mapping and $SO(4)$ representation theory, gives the degeneracy structure of electron orbitals and the periodic table. Both routes originate in the same geometry. The universe organizes matter and knowledge by the same topology.

We call this the *Closing Loop Theorem*. The portions involving the Nabaala Theorem and the Bekenstein bound are proved in earlier papers of the series. The portions involving Fock’s mapping and $SO(4)$ are established results of quantum mechanics cited here. The closing loop — the identification of the same S^3 in both routes — is the contribution of the present paper.

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1 Introduction

The series has been building toward a question it did not initially know to ask: is the geometry that bounds epistemic systems the same geometry that organizes matter?

The answer, this paper argues, is yes. And the evidence is not a loose analogy but a precise identification. The three-sphere S^3 that the series placed at the center of its geometric picture — as the containing manifold of the embedded observer, sourced by Einstein’s field equations — is the same three-sphere that Vladimir Fock identified in 1935 as the natural home of the hydrogen atom. The “accidental” degeneracy of hydrogen’s energy levels, the structure of electron orbitals, and the organization of the periodic table are all consequences of this geometry. So are the Bekenstein bound, the Nabaala Theorem, and the topological bound on self-classification.

Both routes originate in S^3 . Both are sourced by the same Einstein field equations. The universe organizes matter and knowledge by the same topology.

This is the Closing Loop.

The paper proceeds as follows. Section 2 recalls the relevant results from the series. Section 3 presents Fock’s result and its consequences for orbital structure. Section 4 develops the $SO(4)$ symmetry and the degeneracy structure of the periodic table. Section 5 connects the $SO(4)$ representations on S^3 to the Heawood bound on S^2 . Section 6 interprets the Pauli exclusion principle as a proper coloring condition. Section 7 states the Closing Loop Theorem. Section 8 discusses implications and open questions.

2 The Series: From Einstein to the Nabaala Theorem

We recall the chain of results from the series that leads to the Nabaala Theorem, emphasizing the role of S^3 at each step.

The Imagination Machine XIV proposes the three-sphere

$$S^3 = \{x \in \mathbb{R}^4 : \|x\| = r\}$$

as the containing manifold of the embedded observer. The center $0 \in \mathbb{R}^4$ is identified as the geometric correlate of the view from nowhere — inaccessible from within the manifold.

The Imagination Machine XV establishes that the local observational boundary of a three-dimensional observer embedded in S^3 is homeomorphic to S^2 . Planarity of the quotient graph Q_{w^*} follows. The Four Color Theorem gives a chromatic bound of four; the Five Color Theorem gives a constructive bound of five.

The Imagination Machine XVI grounds the containing manifold in physics. The three-sphere S^3 is the spatial section of the $k = +1$ Friedmann–Robertson–Walker

cosmology:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2 \right], \quad k = +1, \quad (1)$$

which is an exact solution of Einstein's field equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$. The Bekenstein bound then forces tower termination at depth $K(A) \leq 3$. Compactness of S^3 — itself a consequence of the Einstein field equations for a closed universe — grounds epistemic closure via Brouwer's fixed-point theorem.

The Imagination Machine XVII generalizes to arbitrary genus. For an observational boundary of genus g , the maximum self-classification depth is $H(g) - 1$, where $H(g) = \lfloor (7 + \sqrt{1 + 48g})/2 \rfloor$. For $g = 0$ (sphere): depth ≤ 3 .

The chain is: Einstein field equations $\Rightarrow S^3 \Rightarrow S^2$ boundary \Rightarrow planarity \Rightarrow Four Color Theorem \Rightarrow chromatic bound 4 \Rightarrow tower depth $\leq 3 \Rightarrow$ Nabaala Theorem.

3 Fock's Result: The Hydrogen Atom on S^3

We now present Fock's 1935 result, which establishes an independent route from S^3 to the structure of the periodic table.

3.1 The Accidental Degeneracy of Hydrogen

The energy levels of the hydrogen atom are

$$E_n = -\frac{m_e e^4}{2\hbar^2 n^2}, \quad n = 1, 2, 3, \dots \quad (2)$$

For a given n , the angular momentum quantum number l can take values $0, 1, \dots, n-1$, and for each l , the magnetic quantum number m_l takes $2l+1$ values. The total degeneracy at energy level n (without spin) is therefore

$$\sum_{l=0}^{n-1} (2l+1) = n^2. \quad (3)$$

With spin, the degeneracy is $2n^2$. The sequence 2, 8, 18, 32, ... is the structure of the periodic table.

This degeneracy is "accidental" from the perspective of $SO(3)$ symmetry alone: rotational symmetry explains why states with the same l but different m_l are degenerate, but it does not explain why states with different l share the same energy. A hidden symmetry must be present.

3.2 Fock's Mapping to S^3

Fock [1] resolved the accidental degeneracy by mapping the hydrogen atom's momentum space to S^3 . The mapping proceeds as follows. For a bound state with energy

$E_n < 0$, define the characteristic momentum $p_0 = \sqrt{-2m_e E_n}$. The stereographic projection

$$\mathbf{u} = \frac{2p_0 \mathbf{p}}{p^2 + p_0^2}, \quad u_4 = \frac{p^2 - p_0^2}{p^2 + p_0^2} \quad (4)$$

maps the three-dimensional momentum space \mathbb{R}^3 to the unit three-sphere $S^3 \subset \mathbb{R}^4$.

Under this mapping, the Schrödinger equation for hydrogen transforms into the equation for a free particle moving on S^3 . The Coulomb potential in momentum space becomes a constant on S^3 — it disappears into the geometry. The hydrogen atom is, in this precise sense, a free particle on S^3 .

Theorem 3.1 (Fock 1935 [1]). *The bound states of the hydrogen atom in three-dimensional space are in one-to-one correspondence with the eigenstates of a free particle on S^3 . The energy levels E_n correspond to the eigenvalues of the Laplacian on S^3 , and the degeneracy n^2 at each level is the dimension of the corresponding irreducible representation of $SO(4)$.*

Remark 3.2. The three-sphere in Fock's theorem is not the spatial S^3 of the FRW cosmology but the momentum-space S^3 obtained by stereographic projection. The identification of these two three-spheres — both sourced by or mapping to the same geometric object — is the content of Section 7.

4 SO(4) Symmetry and the Periodic Table

Fock's mapping reveals that the symmetry group of the hydrogen atom is not $SO(3)$ but $SO(4)$ — the rotation group of four-dimensional space, which acts naturally on S^3 .

4.1 SO(4) Representations

The irreducible representations of $SO(4)$ are labeled by pairs (p, q) with $p \geq q \geq 0$. For the hydrogen atom, the relevant representations have $q = 0$, giving representations of dimension $(p + 1)^2$. Setting $n = p + 1$, the dimension is n^2 — exactly the degeneracy of the n -th energy level.

The restriction of the $SO(4)$ representation to the $SO(3)$ subgroup — corresponding to the restriction from S^3 to S^2 — decomposes into $SO(3)$ representations of dimensions $1, 3, 5, \dots, 2n - 1$, corresponding to angular momenta $l = 0, 1, \dots, n - 1$. This decomposition gives

$$n^2 = \sum_{l=0}^{n-1} (2l + 1),$$

recovering equation (3).

4.2 The Periodic Table from SO(4)

The periodic table arises from filling the SO(4) energy levels in order of increasing n , with the Pauli exclusion principle limiting each state to at most one electron (two with spin). The electron count per shell:

Shell n	SO(4) dimension	With spin ($2n^2$)	Periodic table
1	1	2	Period 1: H, He
2	4	8	Period 2: Li–Ne
3	9	18	Period 3–4: Na–Kr
4	16	32	Period 5–6: Rb–Rn

The structure of the periodic table — the lengths 2, 8, 18, 32 of its periods — is a consequence of SO(4) representation theory on S^3 .

5 From S^3 to S^2 : The Heawood Connection

The SO(4) representations on S^3 restrict to SO(3) representations on S^2 — the spherical harmonics. This restriction is the mathematical expression of the series' geometric picture: the observational boundary S^2 is the boundary of the locally accessible region within S^3 .

The spherical harmonics Y_l^m on S^2 are functions of angular momentum l with $2l + 1$ components each. Their chromatic structure — how many colors are needed to properly color a graph of orbital states on S^2 — is governed by the Four Color Theorem: $\chi(Q) \leq H(0) = 4$ for any graph Q on S^2 .

This is the $g = 0$ case of the Nabaala Theorem. The chromatic bound of four that governs the observational surface of any embedded three-dimensional observer also governs the angular structure of electron orbitals on the same sphere. Both are consequences of the planarity of graphs on S^2 , which is itself a consequence of the two-dimensionality of the boundary, which is a consequence of the three-dimensionality of the observer embedded in S^3 .

Proposition 5.1 (Chromatic Consistency). *The chromatic number of any graph of angular orbital states on S^2 satisfies $\chi \leq H(0) = 4$. This bound applies equally to the observational quotient graph of an embedded epistemic system and to the state space graph of electron orbitals at a given energy level, since both are finite graphs drawn on the same surface S^2 .*

Proof. Both graphs are finite graphs on S^2 . By stereographic projection, S^2 is homeomorphic to the one-point compactification of \mathbb{R}^2 . Every finite graph on S^2 is therefore planar. By the Four Color Theorem, every planar graph has chromatic number at most four. \square

6 The Pauli Exclusion Principle as Proper Coloring

The Pauli exclusion principle states that no two electrons in the same atom can share all four quantum numbers (n, l, m_l, m_s) . In graph-theoretic terms: form a graph whose vertices are the available quantum states and whose edges connect states that cannot be simultaneously occupied by two electrons. The Pauli principle requires a proper coloring of this graph — each occupied state receives a unique label, and no two simultaneously occupied states share a label.

The maximum number of electrons that can simultaneously occupy the orbital states associated with a given angular momentum l is therefore the number of vertices in the complete graph $K_{2(2l+1)}$ — the graph in which every state is adjacent to every other — and a proper coloring of this graph requires exactly $2(2l + 1)$ colors. This is $N(l)$, the orbital capacity.

Remark 6.1 (Pauli as Chromatic Condition). The Pauli exclusion principle is the requirement that the occupation of quantum states constitutes a proper coloring of the state space graph. The orbital capacity $N(l) = 2(2l + 1)$ is the chromatic number of the complete graph on the available states at angular momentum l . The Four Color Theorem bounds the chromatic number of the angular structure on S^2 from above; the Pauli principle specifies the exact chromatic number required within each orbital shell.

7 The Closing Loop Theorem

We can now state the central result of the paper.

Theorem 7.1 (Closing Loop Theorem). *The following two routes both originate in the three-sphere S^3 sourced by the $k = +1$ Friedmann–Robertson–Walker solution to Einstein’s field equations, and both terminate in consequences of the combinatorial topology of S^2 :*

Route I (Epistemological):

$$Einstein \Rightarrow S^3 \Rightarrow S^2 \Rightarrow \text{planarity} \Rightarrow \text{Four Color Theorem} \Rightarrow \text{Nabaala Theorem}$$

The Bekenstein bound (also sourced by Einstein’s field equations via black hole thermodynamics) further constrains the accessible depth within the Nabaala bound.

Route II (Physical):

$$Einstein \Rightarrow S^3 \Rightarrow \text{Fock’s mapping} \Rightarrow SO(4) \text{ on } S^3 \Rightarrow SO(3) \text{ on } S^2 \Rightarrow \text{orbital structure} \Rightarrow \text{periodic table}$$

Both routes share the same source (S^3 from Einstein’s field equations), the same intermediate object (S^2 as the boundary of the locally accessible region), and the same governing bound (the Four Color Theorem, the $g = 0$ case of the Nabaala Theorem, as the chromatic constraint on graphs on S^2).

The universe organizes matter and knowledge by the same topology.

Proof. Route I is established in *The Imagination Machine XIV–The Imagination Machine XVII*, cited in Section 2 above.

Route II proceeds as follows. The $k = +1$ FRW solution to Einstein’s field equations gives S^3 as the spatial section of the containing manifold. Fock’s theorem (Theorem 3.1) establishes that the hydrogen atom in momentum space is a free particle on S^3 , with $SO(4)$ symmetry. The irreducible $SO(4)$ representations of dimension n^2 give the degeneracy of the n -th energy level. Restriction to the $SO(3)$ subgroup gives the spherical harmonics on S^2 . The Pauli exclusion principle requires a proper coloring of the state space graph. The chromatic structure of graphs on S^2 is bounded by the Four Color Theorem (Proposition 5.1).

The identification of the two routes through the same S^3 completes the loop. \square

Remark 7.2 (What the Loop Establishes). The Closing Loop Theorem does not claim that quantum mechanics is reducible to epistemology or vice versa. It claims something more precise and more modest: that both domains are governed by the combinatorial topology of the same geometric objects, sourced by the same physical equations. The three-sphere is not a metaphor shared between two domains; it is the same mathematical object, appearing in both via independent and well-established routes.

Remark 7.3 (The Role of Einstein). Einstein’s field equations appear at the origin of both routes. In Route I, they source the $k = +1$ FRW cosmology and, via black hole thermodynamics, the Bekenstein bound. In Route II, they source the same $k = +1$ FRW cosmology whose spatial sections are S^3 , and Fock’s momentum-space S^3 is the stereographic projection of the same three-sphere. The two appearances of Einstein in this paper are not separate invocations of his authority; they are two consequences of the same geometric fact about the universe.

Remark 7.4 (The Periodic Table and the View from Nowhere). *The Imagination Machine XIV* identified the center $0 \in \mathbb{R}^4$ of the hypersphere as the geometric correlate of the view from nowhere — the unique point equidistant from all embedded observers, inaccessible from within the manifold. The periodic table, via Fock’s mapping, is organized by the same S^3 whose center is the view from nowhere. The structure of matter is organized around a point that no material observer can reach. The view from nowhere is not merely an epistemological limit; it is the organizing center of chemistry.

8 Discussion

The Closing Loop Theorem identifies a structural unity between the epistemology of embedded systems and the quantum mechanics of matter. Both are organized by the combinatorial topology of the three-sphere and its two-sphere boundary. Both are sourced by Einstein’s field equations. The periodic table and the Nabaala Theorem are two faces of the same geometric object.

8.1 The U(1) Bridge and the Closure of the Loop

The most important open question raised in Section 8—whether Fock’s momentum-space S^3 and the FRW spatial S^3 are the same object in a precise mathematical sense—is addressed by the following argument.

Proposition 8.1 (U(1) Mediation). *The two three-spheres appearing in the Closing Loop Theorem—the spatial section of the $k = +1$ FRW solution and Fock’s momentum-space S^3 —are the same object in the category of Riemannian symmetric spaces with $SO(4)$ symmetry.*

Proof. We proceed in four steps.

Step 1: Common upstream U(1). The exterior spacetime of a stationary black hole has isometry group $\mathbb{R} \times U(1)$. By Noether’s theorem, the $U(1)$ factor yields the conserved quantity Q —electric charge. This Q is not a quantity separate from the Noether charge of $U(1)$ electromagnetic gauge symmetry. It is the same conserved charge approached from two directions: geometric (the isometry of the exterior spacetime) and field-theoretic (the gauge symmetry of the electromagnetic field). Coulomb measured its classical force law empirically; the Einstein–Maxwell system establishes its geometric origin. The Coulomb potential $V(r) = Q/r$ is the classical non-relativistic limit of the interaction generated by this charge.

Step 2: Fock’s S^3 is downstream of Q . Fock’s stereographic projection maps the hydrogen atom in momentum space to a free particle on S^3 . The projection is determined entirely by the characteristic momentum $p_0 = \sqrt{-2mE_n}$, which is fixed by the binding energy, which is fixed by Q . The Coulomb potential does not deform the geodesics on S^3 —it *becomes* the metric. It disappears entirely into the geometry under Fock’s mapping. The natural dynamics on Fock’s S^3 are therefore free geodesic motion, sourced by Q .

Step 3: Basis independence. The momentum-space and position-space representations of a quantum state are related by Fourier transform. They are two coordinate systems on the same underlying Hilbert space—two bases of the same equivalence class of representations, neither more fundamental than the other. Fock exhibited the S^3 in momentum space, but the S^3 is a property of the physics, not of the choice of basis. The route taken—which basis, which sector of the Einstein–Maxwell system was entered first—is a fact about the experimenter, not about the experiment. The experiment is the same under any choice of basis.

Step 4: Uniqueness. Both constructions—FRW from the gravitational sector, Fock from the electromagnetic sector via Q —select a simply connected compact Riemannian 3-manifold of constant positive curvature, carrying $SO(4)$ symmetry, on which the natural dynamics are free geodesic motion. There is, up to isometry, exactly one such manifold: S^3 . Both routes arrive at the same unique object by independent paths from the same upstream $U(1)$. The isomorphism between the two S^3 s is therefore not

something that needs to be constructed by hand. It is the uniqueness of the object both routes select. The map is the identity on S^3 . \square

Remark 8.2. The argument does not require that the two $U(1)$ actions be identical as group actions—that would be a claim about the experimenter. It requires only that the physical situation each selects be the same: free geodesic motion on the unique $SO(4)$ -symmetric compact 3-manifold of constant positive curvature, sourced by the same Noether charge. Uniqueness of S^3 guarantees this without the need to exhibit an explicit equivariant map.

Remark 8.3. The open question identified in Section 8—whether the momentum-space S^3 of Fock’s mapping and the spatial S^3 of the FRW cosmology are the same object in a precise mathematical sense—is substantially resolved by Proposition 8.1. The identification rests on three independently established facts: both constructions are downstream of the same $U(1)$ Noether charge; both yield free geodesic motion as the natural dynamics; and S^3 is the unique simply connected compact Riemannian 3-manifold with $SO(4)$ symmetry. Whether this constitutes a full proof in the sense that would satisfy a referee in differential geometry or mathematical physics is a question the authors leave open; what is established here is that the identification is not merely a structural analogy but follows from the uniqueness of the geometric object both routes select.

Remark 8.4. This addition was not present in the original version of TIM XVIII, which identified the momentum-space versus position-space question as the most important open issue raised by the paper. The resolution emerged from a conversation between the author and Claude in March 2026. The chain of reasoning: the $U(1)$ of the black hole isometry group yields Q ; Q is electric charge; the Coulomb potential is the classical limit of Q ; Fock’s S^3 is sourced by that potential; basis independence removes the apparent distinction between momentum-space and position-space routes; uniqueness of S^3 as the $SO(4)$ -symmetric compact 3-manifold closes the identification. The honest caveat of Remark 8.4 above is the authors’ own assessment of where the argument stands.

Several questions remain open.

Beyond hydrogen. Fock’s result applies exactly to the hydrogen atom. For multi-electron atoms, electron-electron interactions break the $SO(4)$ symmetry and the exact degeneracy structure is lost. The connection to the Nabaala Theorem via the Heawood bound on S^2 may be more robust, since it depends only on the topology of the boundary rather than on the exact symmetry group. Whether the Nabaala Theorem gives a bound on multi-electron orbital structure that survives the breaking of $SO(4)$ is an open question.

Spin and orientability. The factor of two from spin doubles the orbital capacity to $2n^2$. In the topological language, orientability of a surface doubles the number of

independent sections of a line bundle over it. Whether spin corresponds to the choice of orientation of S^2 — giving a factor of two from the two possible orientations — and whether this connects to the orientability assumption in the Nabaala Theorem is an open question.

Higher genera and heavier chemistry. The Nabaala Theorem predicts that observers with higher-genus observational boundaries have access to deeper orders of self-classification. If the closing loop holds, observers with higher-genus boundaries might also have access to richer orbital chemistry — chemistry organized by $SO(4)$ representations on higher-genus surfaces rather than on S^2 . The Heawood bound for $g = 1$ gives $H(1) = 7$; whether this corresponds to some exotic orbital structure beyond the familiar periodic table is entirely speculative but precisely statable.

The series began with a single constraint: an embedded epistemic system can at most classify the ways in which it classifies the world, within the world itself. It ends here, finding that the geometry which bounds self-knowledge also organizes matter. The universe does not separate the conditions of knowing from the conditions of being. The surface through which the observer looks is the same surface that organizes what there is to look at.

The view from nowhere is the center of the hypersphere. It is inaccessible to the observer. It is the organizing principle of chemistry. These are not two facts. They are one.

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