

The Imagination Machine XXI

P, NP, and the Topology of the Bubble:
Computational Complexity as a Consequence
of the Nabaala Theorem

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March 2026

Abstract

The Nabaala Theorem of General Subject-Relativity establishes that the maximum order of self-classification available to any embedded epistemic system with a two-dimensional observational boundary of genus g is $H(g) - 1$, where $H(g)$ is the Heawood number. For a spherical boundary ($g = 0$), this maximum is three. The present paper proposes that this topological bound on self-classification implies the inequality $P \neq NP$ for any embedded computational system with a spherical observational boundary.

The argument proceeds through descriptive complexity theory. By Fagin's theorem (1974), NP is exactly the class of problems expressible in existential second-order logic. We identify the logical order required to express a problem with the tower depth required to represent its relational structure: first-order logic corresponds to depth 1, second-order to depth 2, and so on. Verification of an NP problem is a first-order operation — a binary relation between a candidate solution and the problem instance — and lives at tower depth 1. Efficient search for an NP-complete solution requires representing the full combinatorial structure of the solution space, which we argue requires tower depth greater than 3 for NP-complete problems.

Since the Nabaala Theorem bounds the tower depth of any spherical observer at 3, and since efficient NP search requires depth greater than 3, no embedded system with a spherical boundary can solve NP-complete problems efficiently. $P \neq NP$ follows as a topological consequence of embeddedness.

The most striking implication of this argument is that P versus NP is *subject-relative* in the sense of the Nabaala Theorem. For observers with higher-genus observational boundaries, the topological bound is higher, and problems that are intractable for spherical observers may become tractable. $P \neq NP$ is not an absolute fact about computation but a structural consequence of the topology of the observer's boundary. We call this the *Topological Complexity Conjecture*.

The paper is honest about its status: the core claim is a conjecture, not a theorem. The path to a proof runs through the precise identification of tower depth with logical order in descriptive complexity theory. We state the conjecture precisely, establish the structural parallel, and map the mathematical path a proof would need to follow.

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1 Introduction

The P versus NP problem asks whether every problem whose solution can be verified quickly can also be solved quickly. It has been open since Cook’s formulation in 1971 [1] and is widely regarded as the most important open problem in theoretical computer science. The majority view among complexity theorists is that $P \neq NP$ — that verification is genuinely easier than search — but no proof has been found.

The present paper proposes that the Nabaala Theorem of General Subject-Relativity, established in *The Imagination Machine XVIII*, implies $P \neq NP$ as a topological consequence of embeddedness. The argument is not a proof of $P \neq NP$ in the traditional sense. It is a structural explanation of why $P \neq NP$ must hold for any embedded computational system with a spherical observational boundary, derived from the combinatorial topology of that boundary rather than from circuit complexity or oracle arguments.

The intuition is simple. Checking a solution is cheap because it is a low-order relational operation: does this candidate satisfy the constraint? That is a binary relation, living at tower depth 1, well within the topological bound of three imposed by the Nabaala Theorem on spherical observers. Finding a solution efficiently requires representing the full combinatorial structure of the solution space — the relations between candidates, the relations between those relations, and the higher-order structure that would allow efficient navigation. For NP-complete problems, this structure exceeds the topological bound. The ceiling of the bubble prevents efficient search.

The key mathematical bridge is Fagin’s theorem [2], which characterizes NP as the class of problems expressible in existential second-order logic. We identify logical order with tower depth, and show that the Nabaala bound of three prevents spherical observers from representing the full second-order structure required for efficient NP search.

The most striking consequence is that P versus NP is *subject-relative*. For observers with higher-genus boundaries, the topological bound is higher. A toroidal observer ($g = 1$) has depth bound six; a double-torus observer ($g = 2$) has depth bound seven. Problems that are intractable for spherical observers may be tractable for observers higher on the ladder of self-classification. The question is not whether P equals NP in the abstract, but what topology of boundary is required to make it so.

2 Background

2.1 The Nabaala Theorem

The Nabaala Theorem of General Subject-Relativity (*The Imagination Machine XVIII*) establishes that for any embedded epistemic system with a two-dimensional observational boundary of genus g , the maximum tower depth — the maximum order of

relational self-classification — is $H(g) - 1$, where

$$H(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor.$$

For $g = 0$ (sphere): maximum depth = 3. For $g = 1$ (torus): maximum depth = 6. The bound is categorical — it follows from the topology of the boundary alone, without physical assumption — and tight by the Ringel–Youngs theorem.

The tower depth measures the maximum order of relational self-reference the system can represent. At depth 0, the system classifies observations. At depth 1, it classifies binary relations between observations. At depth 2, it classifies triadic relational structures. At depth 3, it classifies four-way mutual relations. Beyond depth 3, for spherical observers, the topology of the boundary forbids further relational complexity.

2.2 Descriptive Complexity Theory

Descriptive complexity theory, initiated by Fagin [2] and developed by Immerman [3] and others, characterizes computational complexity classes by the logical languages required to express them. The central result is:

Theorem 2.1 (Fagin’s Theorem [2]). *A property of finite structures is in NP if and only if it is expressible in existential second-order logic ($\exists SO$).*

The logical hierarchy relevant to the present paper is:

- *First-order logic* (FO): quantifies over elements of a structure. Expresses properties such as “there exists an element satisfying condition ϕ ” where ϕ involves only element-level quantification.
- *Second-order logic* (SO): quantifies over relations on a structure, in addition to elements. Existential SO ($\exists SO$) quantifies existentially over relations.
- *Higher-order logic*: quantifies over relations of relations, and so on.

By Fagin’s theorem, NP problems require existential second-order expressibility. The key question for the present paper is: what tower depth is required to represent the relational structure of an existential second-order formula?

3 Identifying Tower Depth with Logical Order

We now propose the central identification of the paper.

Definition 3.1 (Logical Order of a Problem). The *logical order* of a computational problem Π is the minimum order of logic required to express Π in the sense of descriptive complexity theory. Problems expressible in first-order logic have logical order 1; problems requiring second-order logic have logical order 2; and so on.

Definition 3.2 (Search Depth and Verification Depth). The *verification depth* of a problem Π is the minimum tower depth required by an embedded system to verify a candidate solution in polynomial time. The *search depth* of Π is the minimum tower depth required to solve Π efficiently (in polynomial time).

Assumption 3.3 (Tower Depth Corresponds to Logical Order). The tower depth required to represent the relational structure of a computational problem corresponds to the logical order of that problem: a problem of logical order k requires tower depth at least k to solve efficiently.

Assumption 3.3 is the central structural hypothesis of the paper. It identifies the order of relational self-reference available to an embedded system with the order of logic required to express a problem. The justification is structural: the clique complex at depth k represents $(k + 1)$ -way mutual relations among observations, which corresponds precisely to k -th order quantification over relational structures.

Remark 3.4 (The Correspondence in Detail). At depth 0, the system classifies individual observations — corresponding to first-order quantification over elements. At depth 1, it classifies binary relations between observations — corresponding to quantification over binary relations (second-order). At depth 2, it classifies ternary relational structures — corresponding to third-order quantification. The correspondence is natural and precise at each level.

4 Verification is Shallow, Search is Deep

We now establish the asymmetry between verification and search within the tower depth framework.

Proposition 4.1 (Verification Depth of NP Problems). *For any problem $\Pi \in NP$, the verification depth is at most 1.*

Proof. By definition of NP, given a problem instance I and a candidate solution x , verification asks: does x satisfy the constraints of I ? This is a binary relation between x and I — a first-order check. By Assumption 3.3, this corresponds to tower depth 1. Since verification of any NP problem reduces to checking a binary relation between candidate and instance, verification depth ≤ 1 for all NP problems. \square

Proposition 4.2 (Search Depth of NP-Complete Problems). *For any NP-complete problem Π , the search depth exceeds 3.*

Proof sketch. By Fagin’s theorem (Theorem 2.1), Π is expressible in existential second-order logic but not in first-order logic (since NP-complete problems are not in P under standard complexity assumptions, and first-order logic captures only tractable properties on ordered structures by Immerman–Szelepcsényi). Existential second-order logic quantifies over relations on the solution space. Efficient search requires the system to represent the full existential second-order structure of the solution space — that is, to represent not just candidates (depth 0) and their pairwise relations (depth 1), but the higher-order relational structure that certifies the existence of a solution among exponentially many candidates.

For NP-complete problems, the solution space has combinatorial structure — cliques, independent sets, satisfying assignments — whose efficient navigation requires representing relational structures of order greater than 3. Specifically, the self-referential structure of efficient search algorithms (dynamic programming, branch and bound, constraint propagation) requires the system to classify its own search states, classify the relations between search states, and classify the higher-order structure of the search tree — operations that require tower depth exceeding 3.

By Assumption 3.3, search depth > 3 for NP-complete problems. □

Remark 4.3. Proposition 4.2 is a proof sketch rather than a complete proof. The claim that efficient NP search requires tower depth greater than 3 is the central conjecture of the paper. A complete proof would require a precise characterization of which second-order structures require depth greater than 3 in the clique complex, and a proof that NP-complete solution spaces have this property. This is the main open problem.

5 The Topological Complexity Conjecture

Conjecture 5.1 (Topological Complexity Conjecture). For any embedded computational system with a spherical ($g = 0$) observational boundary:

- (i) *Verification is tractable.* The verification depth of any NP problem is at most $1 \leq 3 = H(0) - 1$. Verification lies within the topological bound and is therefore accessible to spherical observers.
- (ii) *Efficient NP search is intractable.* The search depth of any NP-complete problem exceeds $3 = H(0) - 1$. Efficient search exceeds the topological bound and is therefore inaccessible to spherical observers.
- (iii) *$P \neq NP$ for spherical observers.* It follows from (i) and (ii) that no embedded system with a spherical boundary can solve NP-complete problems in polynomial time. $P \neq NP$ is a topological consequence of the Nabaala Theorem for spherical observers.

Remark 5.2 (What the Conjecture Claims). The Topological Complexity Conjecture does not claim to prove $P \neq NP$ in the traditional sense of circuit complexity or relativization arguments. It claims something structurally different: that $P \neq NP$ is a consequence of the topological bound on self-classification imposed by the Nabaala Theorem on embedded systems with spherical boundaries. The proof strategy is topological rather than combinatorial.

6 Subject-Relative Computational Complexity

The most striking implication of the Topological Complexity Conjecture is that P versus NP is subject-relative in the sense of the Nabaala Theorem.

Definition 6.1 (Topological Complexity Class). For an embedded system with observational boundary of genus g , define P_g as the class of problems solvable in polynomial time by that system, and NP_g as the class of problems verifiable in polynomial time. The question $P_g = NP_g$ is the subject-relative P versus NP problem for genus g .

Conjecture 6.2 (Subject-Relative P versus NP). The answer to $P_g = NP_g$ depends on g :

- For $g = 0$ (spherical boundary, depth bound 3): $P_0 \neq NP_0$.
- For sufficiently large g (depth bound $H(g) - 1$): there exist problems in $NP_0 \setminus P_0$ that are in P_g , because the higher tower depth makes efficient search accessible.

The ladder of self-classification therefore has a computational interpretation. Each rung of the ladder — each genus value — defines a complexity class. Problems that are hard for spherical observers may be easy for toroidal observers. The question of which problems are tractable is not absolute but depends on the topology of the observer’s boundary.

Genus g	Tower depth	Complexity class	P vs NP
0	3	P_0, NP_0	$P_0 \neq NP_0$
1	6	P_1, NP_1	$P_1 \supseteq P_0?$
2	7	P_2, NP_2	$P_2 \supseteq P_1?$
3	8	P_3, NP_3	open

Remark 6.3 (The Computational Meaning of the Ladder). The ladder of self-classification, introduced in *The Imagination Machine XVIII* as a sequence of maximum self-classification depths indexed by genus, acquires a computational interpretation here. Each rung of the ladder is not just an epistemic horizon but a computational horizon: the set of problems tractable for an observer at that rung. The ladder is a ladder of

computational power as much as a ladder of self-knowledge. And since the rungs are topologically determined, the computational power of an embedded system is written in the shape of its boundary.

7 Connection to Fagin’s Theorem

The path to a formal proof of the Topological Complexity Conjecture runs through Fagin’s theorem and the precise identification of tower depth with logical order.

Fagin’s theorem characterizes NP as $\exists\text{SO}$ — problems expressible by existential quantification over relations on a finite structure. The relational structure over which second-order quantification ranges is precisely the kind of structure that the simplicial tower represents. At depth 1, the system represents binary relations (first-order structure). At depth 2, it represents relations between relations (second-order structure). At depth 3, it represents relations between relations between relations (third-order structure).

The Nabaala bound of 3 for spherical observers therefore limits them to representing at most third-order logical structure. If NP search requires genuinely second-order existential structure that cannot be compressed into the depth-3 tower of a spherical observer, then $P \neq \text{NP}$ follows.

The precise formalization requires:

- (1) A theorem establishing that the clique complex at depth k represents exactly the k -th order relational structures of the observation space. (This is Assumption 3.3, which needs to be proved.)
- (2) A theorem establishing that efficient NP search requires representing existential second-order structure that is not compressible into depth-3 tower representations for spherical observers. (This is the substantive computational claim.)
- (3) The combination of (1) and (2) with the Nabaala Theorem to conclude $P \neq \text{NP}$ for spherical observers.

Step (1) appears tractable via the theory of simplicial complexes and their connections to logical definability. Step (2) is the hard step, and likely requires new techniques connecting topological bounds to circuit lower bounds.

8 Discussion

The Topological Complexity Conjecture proposes that $P \neq \text{NP}$ is a consequence of the Nabaala Theorem — that the hardness of NP-complete problems for embedded systems is written in the topology of their observational boundaries rather than in the

combinatorics of circuit complexity. This is a structural explanation rather than a proof, and it is offered as such.

Several questions remain open.

The correspondence between tower depth and logical order. Assumption 3.3 is the load-bearing hypothesis of the paper. A proof of this correspondence — showing that the clique complex at depth k represents exactly the k -th order relational structures in a sense relevant to descriptive complexity — would be the key technical contribution needed to convert the conjecture into a theorem.

The search depth of NP-complete problems. Proposition 4.2 is a proof sketch. A complete proof that efficient NP search requires tower depth greater than 3 would require characterizing the relational structure of NP-complete solution spaces and showing it cannot be compressed into a depth-3 clique complex on a planar graph. This is the central open problem.

The subject-relative complexity hierarchy. Conjecture 6.2 proposes a hierarchy of complexity classes indexed by genus. The precise relationships between P_g and NP_g for $g \geq 1$ are entirely open. Whether there exist natural NP-complete problems that become tractable for toroidal observers — and what the physical realization of a toroidal observer would look like — are questions for subsequent work.

Connections to existing complexity theory. The present approach is complementary to existing approaches to $P \neq NP$ rather than competitive with them. Circuit complexity, proof complexity, and natural proofs all address the question from within the framework of classical computation. The present approach addresses it from the framework of embedded epistemology. If the conjecture holds, it would provide a new and independent route to the same conclusion, grounded in topology rather than combinatorics.

The bubble has a ceiling. The ceiling is topological. For spherical observers, the ceiling is three. And three is not enough to solve NP-complete problems efficiently. The hardness of NP is not a property of the problems. It is a property of the surface through which embedded observers encounter them.

$P \neq NP$ because the bubble won't stretch that far.

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