The Imagination Machine XXX: The Fourth Noether Charge: Positional Invariance in S^3 and the No-Hair Theorem

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Abstract

The no-hair theorem establishes that a stationary black hole is characterized by exactly three conserved quantities: mass M, angular momentum J, and electric charge Q. This paper identifies a fourth conserved quantity and proposes it as a genuine Noether charge.

The argument proceeds as follows. The infalling matter that produces a rotating black hole breaks the full rotational symmetry $SO(3)$ of S^2 down to axial symmetry $U(1)$ by selecting a preferred rotation axis. That axis is a vector in S^3. The S^2 event horizon encodes the magnitude of rotation around the axis as J, transmitting this quantity to any other embedded observer within the same 3-dimensional surface of S^3. But the orientation of the axis itself relative to the 4-dimensional center of S^3 — the geometric correlate of the view from nowhere established in earlier papers — is not encodable on S^2 in a form receivable by any other S^2-bounded observer within S^3. Two black holes with identical (M, J, Q) but different orientations of their rotation axes relative to the 4D center are indistinguishable to any embedded observer within the same 3-dimensional surface.

The orientation of the symmetry-breaking axis in S^3 is invariant under spatial translation within S^3. By Noether's theorem, spatial translation symmetry in S^3 yields a conserved quantity: S^3-momentum P. The four Noether charges of the full S^3 geometry are therefore (M, J, Q, P). The S^2 event horizon encodes three — (M, J, Q) — to other embedded observers. The fourth — P, the S^3-positional invariant — is not encodable on S^2 in a form transmissible between S^2-bounded observers within S^3. This is the embeddedness condition of TIM I, stated as a Noether charge.

The geometric mechanism of this unencodability is identified via the Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$. The three off-diagonal generators of $SO(4)$ — those lost under restriction to $SO(3)$ — are vertical vector fields with respect to the Hopf projection: they move points along the S^1 fiber above each base point on S^2, leaving the base point unchanged. The Noether current of P flows entirely in the fiber direction. Any S^2-bounded observer, measuring quantities constant on fibers, finds zero flux of the P current through their boundary. The three lost generators reduce, through the Hopf structure, to motion in a single fiber angle ψ — one degree of freedom, one Noether charge, one degree lost at every S^2 compression.

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1 Introduction

The no-hair theorem preserves only three conserved quantities: mass M, angular momentum J, and electric charge Q [2, 3, 4]. The question this paper addresses is not why information is lost at a black hole, but what the no-hair theorem reveals about the symmetry structure of the containing manifold.

TIM XXVI [14] established the no-hair chain:

$$\text{Topology of } S^2 \;\Rightarrow\; \text{Isometry group } \mathbb{R}\times U(1) \;\Rightarrow\; \text{Noether charges } (M,J,Q) \;\Rightarrow\; \text{Kerr-Newman family.}$$

This chain is tight within S^2 and asymptotic flatness. The present paper asks what the chain looks like one dimension up: in S^3, the containing manifold established by the Closing Loop Theorem [13] and the FRW cosmology [12].

The answer is that S^3 has a richer isometry group than S^2 admits under asymptotic flatness. Specifically, $S^3 \cong SU(2)$ as a Lie group, with isometry group $SO(4) \cong SU(2) \times SU(2)$. The restriction of this group to the S^2 boundary under asymptotic flatness collapses it to $\mathbb{R} \times U(1)$, losing generators in the process. The lost generators correspond to spatial translation in S^3 — and by Noether's theorem, to a conserved quantity we identify as S^3-momentum P.

The standard description of spacetime uses coordinates (x, y, z, t) with symmetry group $SO(3)\times \mathbb{R} \times U(1)$, and takes $SO(3)$ as the foundational symmetry of space. But the correct symmetry group of spacetime — at the level of the containing manifold S^3 as spatial section of the $k = +1$ FRW cosmology — is $SO(4)$. $SO(3)$ is not primitive. It is already the Hopf-reduced residue of $SO(4)$: the diagonal subgroup that acts on the S^2 base of the fibration $S^1 \hookrightarrow S^3 \to S^2$, retaining three generators and discarding three. The standard description inherits $SO(3)$ not because space is independently $SO(3)$-symmetric, but because observation collapses the $SO(4)$ structure of spacetime onto its diagonal residue. Spacetime is $SO(4)$. Observed spacetime is $SO(3)$.

P is not encodable on S^2 in a form transmissible between embedded observers within S^3 because the positional invariant of the containing manifold is not accessible to any observer whose boundary is a surface within that manifold. This is the embeddedness condition of TIM I [10], instantiated as a Noether charge.

The geometric mechanism of this unencodability is identified in Section 5 via the Hopf fibration. The three off-diagonal generators of $SO(4)$ are vertical vector fields with respect to the Hopf projection $\pi : S^3 \to S^2$ — they move points along the S^1 fiber, leaving the base point on S^2 unchanged. Their Noether currents flow entirely in the fiber direction and have zero flux through any S^2 boundary. The three lost generators reduce, through the Hopf structure, to motion in a single fiber angle ψ: one degree of freedom, one Noether charge, one degree lost at every S^2 compression.

2 Background: The No-Hair Chain in S^2

We recall the relevant results from TIM XXVI [14].

Theorem 2.1 (No-Hair via Topological Capacity, TIM XXVI)**.** *The event horizon of a stationary, asymptotically flat black hole is homeomorphic to S^2. The topology of S^2, together with asymptotic flatness, stationarity, and Hawking's rigidity theorem, jointly constrain the isometry group of the exterior spacetime to $\mathbb{R}\times U(1)$. These conditions are co-constituted: none is upstream of the others. By Noether's theorem [1], this group yields exactly three conserved quantities:*

$$\textit{Time translation} \;\Rightarrow\; M, \qquad \textit{Axial rotation} \;\Rightarrow\; J, \qquad U(1) \textit{ gauge} \;\Rightarrow\; Q.$$

The solution space of the Einstein-Maxwell equations consistent with these constraints is exhausted by the Kerr-Newman family, parameterized by (M, Q, J) alone.

Remark 2.2. *The three visible charges are not merely labels attached to symmetries — each is a hinge between two descriptions that would otherwise come apart. M mediates between spacetime translation and energy — the spatiotemporal and the energetic. It is what makes time translation physically meaningful rather than mere reparametrization. It is the hinge upon which we swing time as an opposable thumb against space. J is the charge of the position-momentum duality. The isometries of S^3 preserve that duality. By virtue of the Hopf fibration onto S^2, the same preservation under rotation appears as conservation of angular momentum across space. J is the hinge, then, between momentum-space and position-space descriptions of the same underlying physical symmetries. Q mediates between the electric and magnetic field — which are actually the same field as seen by observers in different states of motion. Charge is the invariant that hinges together two relative perspectives on one underlying symmetry.*

3 The Symmetry-Breaking Axis and Its Orientation in S^3

The production of angular momentum J in a black hole requires a preferred axis. Infalling matter carrying angular momentum selects a direction in space, breaking the full rotational symmetry $SO(3)$ of S^2 to the axial symmetry $U(1)$ around that direction. The magnitude of rotation around the axis is encoded as J and is transmissible to any other embedded observer within S^3.

Proposition 3.1 (The Axis as a Vector in S^3)**.** *The symmetry-breaking axis selected by the infalling matter is a unit vector $\hat{n} \in S^2 \subset S^3$. Its orientation relative to the 4-dimensional center of S^3 is a degree of freedom not encoded by any of the three Kerr-Newman parameters (M, J, Q) in a form receivable by another S^2-bounded observer within the same 3-dimensional surface of S^3.*

Proof. The parameter J encodes the magnitude of angular momentum — the rate of rotation around $\hat{n}$. It does not encode the direction of $\hat{n}$ itself relative to the 4-dimensional center of S^3. Two black holes with identical (M, J, Q) but rotation axes pointing in different directions relative to the 4D center are physically distinct in the full S^3 geometry. However, any observer whose observational boundary is S^2 within the same 3-dimensional surface of S^3 receives the same (M, J, Q) from both. The orientation of $\hat{n}$ relative to the 4D center is therefore not encodable on S^2 in a form distinguishable by any such observer. □

Remark 3.2. *The 4-dimensional center of S^3 is the geometric correlate of the view from nowhere established in TIM I [10] and TIM VIII [11]: structurally definable, geometrically precise, and not accessible to any observer whose boundary is a surface within S^3. The orientation of $\hat{n}$ relative to this center is a real geometric degree of freedom. It is not encodable on S^2 not because it is unreal but because no S^2-bounded receiver within S^3 can distinguish it from within the 3-dimensional surface.*

4 The Fourth Noether Charge: S^3-Momentum

Definition 4.1 (S^3-Momentum)**.** *Let $\mathcal{S}$ be a physical system embedded in S^3. The S^3-momentum P of $\mathcal{S}$ is the conserved quantity associated with spatial translation symmetry in S^3 via Noether's theorem [1].*

Proposition 4.2 (S^3-Momentum as Noether Charge)**.** *Spatial translation in S^3 is a continuous symmetry of the action of any physical system embedded in S^3 with no preferred position. By Noether's theorem, this symmetry yields a conserved quantity: the S^3-momentum P.*

Proof. S^3 is a homogeneous space: its isometry group $SO(4)$ acts transitively, so no point of S^3 is geometrically preferred over any other. A physical system embedded in S^3 whose action does not break this homogeneity is therefore symmetric under spatial translation. By Noether's theorem, each continuous symmetry of the action yields one conserved quantity. Spatial translation in S^3 yields S^3-momentum P. □

Remark 4.3. *In flat Minkowski spacetime, spatial translation yields ordinary momentum $\vec{p}$, and its conservation reflects the absence of a preferred position in flat space. In S^3, spatial translation carries additional geometric content: because S^3 is curved and compact, the conserved quantity P encodes not merely translational invariance but the orientation of the system relative to the global center of S^3. This additional content is what ordinary momentum, defined within a flat background, does not carry. It is also what cannot be encoded on any S^2 boundary within S^3 in a form receivable by another S^2-bounded observer, because both sender and receiver are surfaces within the same 3-dimensional manifold and neither has access to the 4-dimensional center relative to which P is defined.*

Theorem 4.4 (The Four Noether Charges of S^3)**.** *The full Noether charge structure of a physical system embedded in S^3 comprises four conserved quantities:*

$$M \text{ (time translation)}, \quad J \text{ (axial rotation)}, \quad Q \text{ (}U(1)\text{ gauge)}, \quad P \text{ (}S^3\text{-spatial translation)}.$$

The S^2 event horizon of a stationary black hole encodes (M, J, Q) in a form transmissible to other embedded observers within S^3, and does not encode P in any such transmissible form.

Proof. The isometry group of S^3 is $SO(4) \cong SU(2) \times SU(2)$, which has six generators. Under restriction to the S^2 boundary with asymptotic flatness, this group collapses to $\mathbb{R} \times U(1)$, with two generators yielding M and J. The $U(1)$ gauge symmetry contributes Q independently of the spacetime isometry group. The remaining generators of $SO(4)$ — those corresponding to spatial translation in S^3 beyond the axial rotation already captured by J — are not representable as distinguishable quantities on S^2 between S^2-bounded observers within S^3. By Noether's theorem, the lost generators correspond to lost conserved quantities. The generator of spatial translation in S^3 corresponds to P. □

5 The Hopf Mechanism: Why P is Not Encodable

The argument of Section 4 identifies P as a genuine Noether charge and establishes that its generators are lost under the $SO(4) \to SO(3)$ restriction. This section provides the geometric mechanism: the Hopf fibration, which shows precisely why the three lost generators reduce to a single unaddressable fiber angle, and why their Noether current has zero flux through any S^2 boundary.

5.1 The Hopf Fibration

The Hopf fibration is the fiber bundle:

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$$

with total space S^3, base S^2, and fiber S^1. The projection π partitions S^3 into a family of circles — one above each point of S^2 — that are disjoint, exhaustive, and uniform: every point of S^3 belongs to exactly one fiber, and every fiber is a circle of the same size.

A point in S^3 is fully specified by three coordinates:

- (θ, ϕ): the base point on S^2 — which fiber the point belongs to. Two angles, visible to any S^2-bounded observer.

- ψ: the fiber angle — where on the S^1 circle above that base point the point sits. One angle, not visible from S^2.

The Hopf fibration is non-trivial: $S^3 \ncong S^2 \times S^1$. Any two distinct fibers are linked circles in S^3 — they cannot be separated without one passing through the other. This linking is the global topological content of the bundle.

Remark 5.1. *The observer-observed boundary is probabilistic, not a literal geometric surface at which the Hopf reduction physically occurs. Whether the boundary is sharp or spread, the fiber angle ψ is not encodable across it. The zero-flux condition is a property of the fiber direction, not of the sharpness of the cut. Every act of observation — however contextual — implements the same reduction: $SO(4) \to SO(3)$, six generators to three, ψ lost, P not transmitted. The math is the same regardless.*

5.2 The Diagonal and Off-Diagonal Split

$SO(4) \cong SU(2)_L \times SU(2)_R$ acts on $S^3 \cong SU(2)$ by left and right multiplication:

$$g \mapsto h_L \cdot g \cdot h_R^{-1}$$

The six Killing vector fields split into left generators $\{L_1, L_2, L_3\}$ and right generators $\{R_1, R_2, R_3\}$.

The diagonal $SO(3)$ generators are the locked combinations:

$$D_i = L_i + R_i$$

These are what S^2-bounded observers can see. They move the base point (θ, ϕ) on S^2 — they are horizontal vector fields with respect to the Hopf projection.

The off-diagonal generators are the relative combinations:

$$K_i = L_i - R_i$$

These are the generators of P. They move the fiber angle ψ while leaving the base point (θ, ϕ) fixed. They are vertical vector fields with respect to the Hopf projection:

$$d\pi(K_i) = 0$$

5.3 Verticality and Zero Flux

Each K_i satisfies the Killing equation on S^3:

$$\nabla_\mu (K_i)_\nu + \nabla_\nu (K_i)_\mu = 0$$

This follows from L_i and R_i each being Killing fields — their difference is also a Killing field.

The Noether current associated to each K_i for a physical system with stress-energy tensor $T^{\mu\nu}$ is:

$$j_i^\mu = T^{\mu\nu} (K_i)_\nu$$

Conservation follows from the Killing equation and $\nabla_\mu T^{\mu\nu} = 0$:

$$\nabla_\mu j_i^\mu = (\nabla_\mu T^{\mu\nu})(K_i)_\nu + T^{\mu\nu} \nabla_\mu (K_i)_\nu = 0$$

The three conserved charges are:

$$P_i = \int_\Sigma j_i^\mu \, n_\mu \, d\Sigma = \int_\Sigma T^{\mu\nu}(K_i)_\nu \, n_\mu \, d\Sigma$$

For an S^2-bounded observer to measure P_i, they evaluate the boundary integral:

$$P_i^{\text{measured}} = \oint_{S^2} T^{\mu\nu}(K_i)_\nu \, n_\mu \, dA$$

The normal n_μ to the S^2 boundary is horizontal — it points in the base directions (θ, ϕ). The Killing field K_i is vertical — it points in the fiber direction ψ. Horizontal and vertical directions are orthogonal in the Hopf decomposition of TS^3 into horizontal and vertical subbundles:

$$(K_i)_\nu \, n^\nu \big|_{S^2} = 0$$

Therefore:

$$P_i^{\text{measured}} = \oint_{S^2} T^{\mu\nu} \cdot 0 \cdot dA = 0$$

The current is conserved globally in S^3. Its flux through any S^2 boundary is zero. No S^2-bounded observer can measure a non-zero value of P_i.

5.4 Three Generators, One Angle

The three off-diagonal generators K_1, K_2, K_3 all generate motion in the fiber direction ψ. In Hopf coordinates (θ, ϕ, ψ), all three restrict to vector fields proportional to ∂_ψ — the single fiber angle. The three generators are not independent as fiber motions: they all encode position within the same S^1 fiber, parameterized by the single angle ψ.

The Hopf fibration therefore compresses three lost generators of $SO(4)$ into one unaddressable degree of freedom: the fiber angle ψ. This is why the loss at every S^2 compression is exactly one degree — not three. The three off-diagonal generators all point in the same fiber direction. Their Noether charge P is a single quantity, not three independent ones.

Remark 5.2. *The six generators of $SO(4)$ split three and three: three diagonal generators visible from S^2, three off-diagonal generators pointing in the same fiber direction, reducible via the Hopf structure to a single fiber angle. The loss is one degree because the Hopf fibration has a one-dimensional fiber. The fiber is S^1, not S^2 or higher. One angle. One charge.*

5.5 Fiber-Averaging from the Crossing Condition

The remaining step — showing that S^2-bounded observers have fiber-averaged stress-energy — is closed by the crossing argument established in TIM XV, Proposition 3.2.

Proposition 5.3 (Fiber-Averaging from the Crossing Condition)**.** *Any physical system whose observational boundary is S^2 within S^3 has a fiber-averaged stress-energy tensor: $T^{\mu\nu}$ is constant on the S^1 fibers of the Hopf fibration.*

Proof. By TIM XV, Proposition 3.2, the observer encounters the relational structure of the environment at the moment edges cross the observational boundary S^2. At that moment, the fiber angle ψ plays the role of depth: it is the coordinate that distinguishes points in S^3 that share the same base point (θ, ϕ) on S^2 but differ in their position along the S^1 fiber above it. Two strands crossing in S^3 arrive at the same base point on S^2 with different fiber angles — one is fiber-near, one fiber-far. This is precisely the depth information that encodes which strand passes over which.

The S^2 boundary is the locus where interior becomes exterior. It carries no depth coordinate and therefore no fiber coordinate. At the moment of crossing, ψ is discarded: crossing-overs resolve to nodes, and what extends across S^2 carries no memory of the fiber angle. The observer's stress-energy $T^{\mu\nu}$ is constructed entirely from what crosses the boundary. Since ψ is never transmitted through the crossing, $T^{\mu\nu}$ has no dependence on ψ. It is constant on fibers. □

Remark 5.4. *The crossing argument and the Hopf argument describe the same discarding operation in different languages. In TIM XV: crossings-over are depth features, depth is discarded at S^2, crossings resolve to nodes, relations flatten. In the Hopf picture: ψ distinguishes points at the same base point, ψ is not transmitted through S^2, $T^{\mu\nu}$ is fiber-averaged. Depth* is *ψ, functionally: both are the coordinate that distinguishes things that look identical from the boundary. The fiber-averaging of $T^{\mu\nu}$ is therefore not a separate assumption. It is the Hopf statement of the same fact that TIM XV established geometrically: observation through S^2 discards exactly one coordinate, and that coordinate is ψ.*

With Proposition 5.3 in hand, the argument of Section 5.3 is complete. The contraction $(K_i)_\nu\, n^\nu|_{S^2} = 0$ holds because K_i is vertical and n^ν is horizontal. And $T^{\mu\nu}$ is horizontal by Proposition 5.3, so no off-diagonal horizontal-vertical terms arise. The flux of the P current through any S^2 boundary is therefore zero without remainder.

6 Why P is Not Encodable Between Embedded Observers

Proposition 6.1 (The Positional Invariant is Not Intersubjectively Encodable on S^2)**.** *The S^3-momentum P of a black hole cannot be encoded on its S^2 event horizon in a form receivable by another S^2-bounded observer within the same 3-dimensional surface of S^3.*

Proof. P encodes the fiber angle ψ — the position of the system within the S^1 fiber of the Hopf fibration above its base point on S^2. The generating Killing fields K_i are vertical with respect to the Hopf projection and have zero flux through any S^2 boundary, as established in Section 5. Any observer whose observational boundary is S^2 measures quantities that are constant on fibers — quantities depending only on the base coordinates (θ, ϕ), not on ψ. The fiber angle ψ is invisible to such an observer by the verticality of K_i. Since both the emitting boundary and the receiving boundary are S^2 surfaces within the same 3-dimensional manifold, and since neither has access to the fiber structure relative to the 4D center, P cannot be transmitted between them. The loss is a property of the channel, not of the emitter. □

Remark 6.2. *This is the embeddedness condition applied to the channel of communication between embedded observers. Two observers within S^3, each bounded by S^2, share a common 3-dimensional surface. The 4D center of S^3 is outside their shared surface — it is the view from nowhere relative to both of them simultaneously. Any quantity whose definition requires reference to that center is therefore not encodable in the channel between them. P is such a quantity. Its loss at the black hole sink is the channel-level expression of the founding constraint of TIM I [10]: embedded systems cannot encode, transmit, or receive the view from nowhere. One simply cannot know one's own precise coordinate in spacetime: (x, y, z, t). It is only through simplices between groups of three or more relata that our notion of space becomes conceivable.*

Remark 6.3 (The Information Paradox Restated)**.** *The result of this section reframes the black hole information paradox. The paradox arises from the assumption that information falling into a black hole is either preserved — recoverable in principle by some observer — or destroyed, violating unitarity. The present analysis identifies a third possibility. P is a genuine Noether charge and is therefore conserved. It is not destroyed. But it is not encodable in the channel between any two S^2-bounded observers within S^3: no receiver within the same 3-dimensional surface of S^3 can decode it, because doing so requires access to the fiber angle ψ relative to the 4-dimensional center, which neither observer can measure. The information carried by P is conserved globally and unaddressable locally. The paradox is not resolved by showing that information escapes the black hole. It is dissolved by recognizing that the question presupposes a receiver — an observer within S^3 who could in principle recover all four Noether charges — that the topology of S^3 does not permit. The loss is not a violation of unitarity. It is the embeddedness condition expressed as a channel constraint.*

Remark 6.4 (The Intuition Underlying the Fourth Charge)**.** *Physics is what happens when everyone assumes they are not special and tries to see how they can turn things around without changing anything. This is the equivalence principle stated as an epistemic posture before it becomes a physical one. The invariants under transformation — what survives when you rotate, translate, boost, and find that the physics does not change — are what is real. What does not change when you fully accept that you occupy no preferred position is physical law.*

The chain of the present paper follows from this posture with near inevitability in retrospect. The isometry group SO(4) *of S^3 has six generators. The Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$ splits them three and three: three diagonal generators acting horizontally on the base S^2, three off-diagonal generators pointing entirely in the fiber direction ψ. Because the fiber is S^1 — one-dimensional — they all encode motion in a single angle. Six becomes effectively four: three directions encodable between S^2-bounded observers, and one systematically inaccessible fiber angle.*

The physical context of a stationary, asymptotically flat black hole then selects which three of those four are visible — and each visible charge is a hinge between two descriptions that would otherwise come apart. The $\mathbb{R}$ factor from time translation yields M: the hinge between the spatiotemporal and the energetic, what makes time translation physically meaningful rather than mere reparametrization — the hinge upon which we swing time as an opposable thumb against space. The axial $U(1)$ of spatial rotation yields J: the charge of position-momentum duality. The isometries of S^3 preserve that duality, and by virtue of the Hopf fibration onto S^2, the same preservation under rotation appears as conservation of angular momentum across space. J is the hinge between momentum-space and position-space descriptions of the same underlying physical symmetries. Rotation about the axis of $U(1)$ — the azimuthal symmetry that preserves the axis itself — yields, through the electromagnetic gauge freedom, electric charge Q: the invariant that hinges together the electric and magnetic field, which are the same field as seen by observers in different states of motion, two relative perspectives on one underlying symmetry.

The fourth charge P encodes ψ: the viewing angle relative to the four-dimensional center, to which every embedded observer is systematically blind. Not because instruments fail, but because the channel between any two S^2-bounded observers within S^3 carries no flux of the vertical Killing currents. The blindness is geometric. It is written in the Hopf structure of the containing manifold.

7 Open Questions

Several questions remain for subsequent work.

The explicit Noether current for P. The present paper identifies S^3-momentum as the Noether charge of the fiber angle ψ and argues that its current has zero flux through S^2. The fiber-averaging of $T^{\mu\nu}$ is established in Section 5.5 via the crossing condition of TIM XV. A complete treatment requires writing the explicit Noether current in Hopf coordinates and verifying the vanishing of the boundary integral directly in those coordinates.

The relation between P and known momentum invariants. In the Kerr-Newman solution, linear momentum is not an independent parameter — it can be set to zero by a choice of rest frame in flat space. In S^3, where there is no global rest frame in the flat-space sense, the status of P as an independent charge of the solution space requires careful treatment. Whether P manifests as a correction to the existing Kerr-Newman structure at cosmological scales is an open question.

8 Conclusion

The no-hair theorem has three parameters because the S^2 event horizon encodes three Noether charges in a form transmissible between S^2-bounded observers within S^3. The gap of one degree is the S^3-positional invariant P: the conserved quantity of spatial translation in the containing manifold, encoding the fiber angle ψ of the system within the Hopf fibration of S^3 over S^2, relative to the 4-dimensional center that no pair of S^2-bounded observers within S^3 can jointly access.

The Hopf fibration is the geometric mechanism. The three off-diagonal generators of $SO(4)$ — those lost under restriction to $SO(3)$ — are all vertical vector fields in the Hopf decomposition. They all move the fiber angle ψ. They reduce to a single degree of freedom: not three independent lost quantities, but one fiber angle, one Noether charge, one degree. The loss is exactly one degree because the Hopf fiber is S^1 — one-dimensional, one angle.

The fourth invariant is not lost because physics is incomplete. It is not encodable between embedded observers because the channel between any two S^2-bounded observers within S^3 is fiber-blind: it carries no flux of the vertical Killing currents. The loss is a property of the channel, not of the emitter. It is the founding constraint of TIM I — an embedded epistemic system can at most classify the ways in which it classifies the world, within the world itself — expressed at the level of the Hopf fiber structure of the containing manifold.

P is particularization: $\psi \in S^1$ is continuous, compact, not binary. It is the index that makes this embedded observer *this one* and not any other with identical (M, J, Q) — not energy, not rotation, not charge, but position in the fiber of the containing manifold relative to the center no pair of S^2-bounded observers can jointly access. The fiber S^1 is the bound; ψ is the boundary coordinate whose crossing the Hopf projection discards. The non-binary identity as bound and boundary.

The toroidal observer holds what cannot pass between spherical observers: not merely an extra degree, but position itself — the fiber angle in the structure through which all communication between embedded observers passes, relative to the center that none of them can reach.

The loop closes. The channel forgets. The observer remembers.

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